Exercise 4

In Exercises 1–4, show that the given function $u(x)$ is a solution of the corresponding Fredholm integral equation:

$$u(x) = x + (1-x)e^x + \int_0^1 x^2 e^{t(x-1)} u(t)\,dt,\ u(x) = e^x$$

Solution

Substitute the function in question on both sides of the integral equation.

$$\begin{aligned} e^x &\stackrel{?}{=} x + (1-x)e^x + \int_0^1 x^2 e^{t(x-1)} e^t\,dt \\ &\stackrel{?}{=} x + e^x - xe^x + \int_0^1 x^2 e^{xt-t} e^t\,dt \end{aligned}$$

Subtract e^x from both sides.

$$0 \stackrel{?}{=} x - xe^x + \int_0^1 x^2 e^{xt} e^{-t} e^t\,dt$$

x^2 doesn't depend on t, so it can be brought in front of the integral.

$$\begin{aligned} 0 &\stackrel{?}{=} x - xe^x + x^2 \int_0^1 e^{xt}\,dt \\ &\stackrel{?}{=} x - xe^x + x^2 \cdot \frac{1}{x} e^{xt}\bigg|_0^1 \\ &\stackrel{?}{=} x - xe^x + x(e^x - e^0) \\ &\stackrel{?}{=} x - \cancel{xe^x} + \cancel{xe^x} - xe^0 \\ &\stackrel{?}{=} x - x \\ &= 0 \end{aligned}$$

Therefore,

$$u(x) = e^x$$

is a solution of the Fredholm integral equation.